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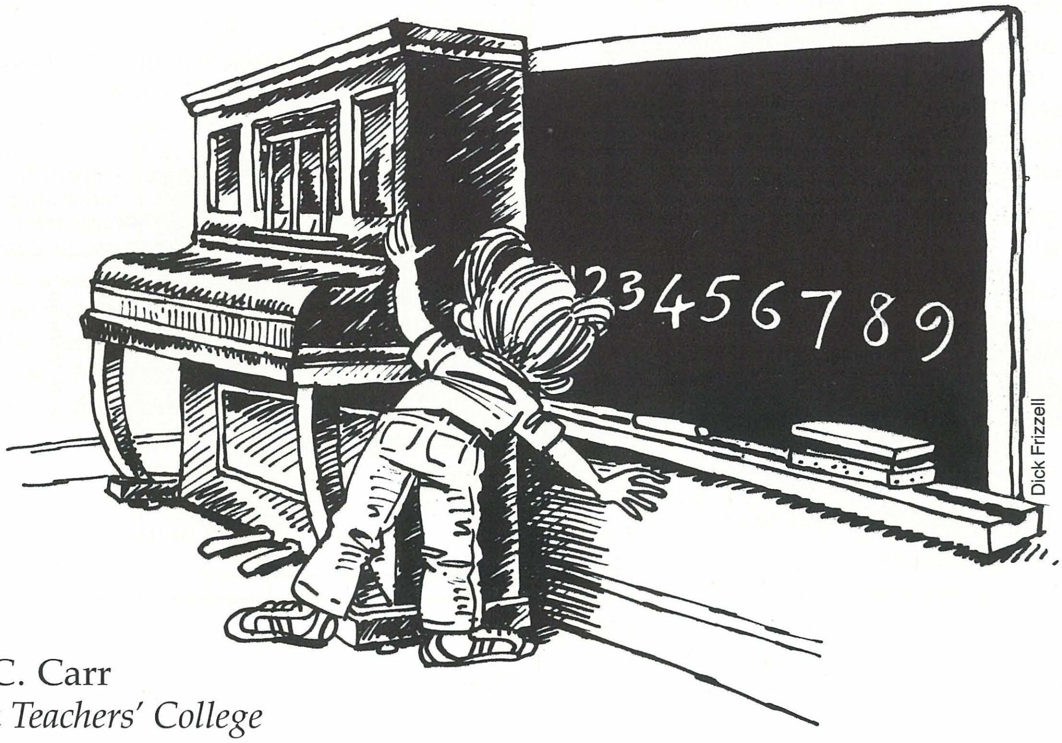
number two 1986

There Are Numbers Behind the Piano

Children's Construction of Meaning in Mathematics



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Introduction

The study of children's mathematical behaviour goes back many years. Piaget in the 1930s used clinical interviews to unearth children's ideas about number. Earlier than this, Brueckner had used tests that measured ability in computation and solving verbal problems, and tests that sampled various combinations of a particular skill (such as the addition of fractions).

In a short time the picture emerged that mathematics was an extremely difficult subject for many children to master. Some people, however, claimed that it was mathematics *education* that was failing.

To investigate the problems, large scale surveys, such as the National Assessment of Educational Progress (NAEP) in the United States of America, have been used. The latest NAEP assessments sampled 45,000 students. In Britain the Concepts in Secondary Mathematics and Science (CSMS) research programme assessed 10,000 students. Likewise, the International Association for the Evaluation of Educational Achievement (IEA) survey in mathematics used in New Zealand a sample of 5177 for the 'core test' as part of a study of secondary school mathematics in 23 countries.

At the other end of the continuum researchers have probed the mathematical ideas of individual children through interviews, in an attempt to uncover underlying cognitive processes. These studies usually focus on one specific topic within mathematics, and provide information on the learner's view of the processes involved.

Still other research adopts both techniques. Brown in 1981 used data from the CSMS survey and expanded upon this with interviews. Brown's study revealed that students in the 11 to 16 age group had considerable gaps in their knowledge in place value and decimal fractions. Brown and Van Lehn investigated the errors, or 'bugs', that students generated when confronted with multidigit subtraction. Van Lehn listed 77 'bugs' (systematic errors) that students made!

Along with research into children's errors in mathematics, have come some studies that emphasise more positive aspects. Moser and Carpenter in 1982, and Gelman and Gallistel in 1983 noted how capable young children are in counting and solving verbal problems. Earlier, Donaldson and her co-workers found that children could conserve number younger than expected, and could understand the relationship between class and sub-class (in a set of objects). Other research revealed that, given alternative approaches, previously mathematically incompetent adults could make appreciable progress in computational skills and understanding.

The Evidence

In an analysis of particular items in the NAEP survey it was found that over half of the 13-year-olds could not calculate the area of a rectangle from its dimensions. Although most could identify common geometric shapes, fewer than 10% could use the knowledge that the sum of the angles of a triangle is 180 degrees to find the measure of the third angle. In other words relatively few students demonstrated knowledge of the basic properties of geometric shapes.

Problems were found to be just as frequent by the British CSMS study. In summarizing the results, Hart concluded that:

> The overwhelming impression obtained is that Mathematics is a very difficult subject for most children.
>
> And we have shown that understanding improves only slightly as the child gets older.
>
> And in the secondary school we tend to believe that the child has a fund of knowledge on which we can build the abstract structure of mathematics. The child may have an amount of knowledge but it is seldom as great as we expected.

Cockcroft in 1982 worked with 107 adults, as described in his report. He said,

> The extent to which the need to undertake even an apparently simple and straightforward piece of mathematics could induce feelings of anxiety, helplessness, fear and even guilt in some of those interviewed was, perhaps, the most striking feature of the study.

The recent IEA survey in New Zealand showed surprising weaknesses by third formers (13-year-olds). For example, only half of the children said 20% = 1/5, the most common error was 5% = 1/5. Fewer than half the 13-year-olds could successfully answer items on common fractions, decimal fractions, estimation of area, assigning points to a number line, and basic algebraic computation.

Some of the most revealing student misconceptions in mathematics have been unearthed by Stanley Erlwanger. One of the most frequently quoted examples comes from his conversation with Benny, an above-average 12-year-old. Benny's teacher, in fact, regarded him as one of her best pupils in mathematics. Benny's procedure for the addition of decimal fractions was as follows:– (E = Erlwanger; B = Benny)

> E: Like, what would you get if you add point 3 and point 4?
> B: That would be... oh seven... Point 07.
> E: How did you decide where to put the point?
> B: Because there's two points; at the front of the 4 and the front of the 3. So you have to have two numbers after the decimal, because... you know... two decimals. Now like if I had point 44, point 44 [i.e., .44 + .44], I have to have four numbers after the decimal [i.e., .0088].

In further exploring Benny's ideas and beliefs about mathematics, Erlwanger discovered that Benny considered that mathematics consisted of different rules for different types of problems. Benny's purpose in learning mathematics seemed to be to discover rules and use these to solve problems. There was only one rule for each type of problem, according to Benny.

During 1982 I studied the progress of eight 12-year-old students who were academically representative of their school class. The class was embarking upon four weeks of work on decimal numbers and decimal fractions and I interviewed the students before and after the work using a series of nine stimulus cards. These cards covered estimation, division with a divisor greater than the dividend, writing decimal numbers, problem solving involving decimal numbers, comparing numbers containing decimal fractions, and naming the place value columns in decimal numbers. These topics matched the instructional objectives for the teaching module. Individual interviews were audio-taped and the tapes transcribed.

Table 1 presents the results. The students are arranged in order of academic achievement – highest being 'Jo'.

Table 1
Response Movement on Stimulus Cards – Before and after Instruction.

Subject:	Correct-remaining-correct	Correct-to-incorrect (Change)	Incorrect-remaining-incorrect	Incorrect-to-correct (Change)
Tim	0	0	9	0
Mary	1	0	8	0
Bob	2	0	5	2
Sarah	4	0	4	1
Bevan	4	0	2	3
Sue	6	2	1	0
Oliver	4	2	1	2
Jo	7	1	0	1
Σ	28	5	30	7

Despite four weeks' work the 'Incorrect-remaining-incorrect' category is as common as the 'Correct-remaining-correct' class. In other words there was little change. However, worse still, there are 5 examples of Correct-to-incorrect. And these are among the more able students Sue, Oliver and Jo.

In attempting to explain the (apparent?) regression, Erlanger and Benny come to mind. Benny often constructed his own (unintended) meaning from the mathematics programme.

Because of the small sample in my study, it would be unwise to generalize from these results. As well, the teacher could be important, although from my observations the particular module of work was carefully planned and well taught by an able teacher. In spite of these limitations, the results do provide additional evidence that for many students slow progress in mathematics is the norm.

Lest we become too pessimistic, research evidence has also pointed out that children are surprisingly competent in mathematics. In particular, young children have more knowledge of the principles of number and counting operations than was previously supposed. For example, children as young as two and a half used the cardinal principle:

> For the child as young as two and a half years, enumeration already involves the realization that the last numerlog in a set (at least in a small set) represents the cardinal number of the set.

Ninety percent of six-year-olds can solve addition verbal problems. About half of them use advanced counting-on procedures, that is, they enter the sequence at a place corresponding to one addend, then count forward as many words as indicated by the second addend in order to reach the answer.

The inventive powers of children in mathematics have been well documented, a good book being *Understanding Mathematics* by R.B. Davis, published in 1984. Teachers sometimes report to their colleagues the intuitive discoveries a child in their class makes. Alan Hall reported one 10-year-old's realization that negative integers could be recorded symbolically – this particular child realized there should be numerals on the number line 'behind the piano', the piano at the front of the classroom was obscuring that part of the number line to the left of zero.

Children attempt to construct some meaning from whatever they confront in mathematics. Sometimes this is the meaning that the teachers intend. At other times the children's active construction produces new errors, or stabilizes existing misconceptons.

A question researchers have asked is what meanings children construct from statements in mathematics that do not make sense? Do children attempt to answer bizarre questions? Children *do* attempt to answer bizarre questions about the world, for example, 'Is Red wider than Yellow?', but is this applied to questions in mathematics?

With the assistance of colleagues, I made up five bizarre questions, set out in Table 2.

Table 2
Bizarre Questions in Mathematics.

1. We are measuring using paces, or strides. It is 50 paces around a truck.
   How heavy would the load be on this truck?
2. It takes one man a day and a half to dig a hole and a half.
   How long will it take two men?
3. There are ten people. Each has five apples.
   Who ate them all first?
4. Some children sat a test. The top mark was 55.
   The bottom mark was 5.
   How many people sat the test?
5. It takes me ten minutes to bike five kilometers.
   How long will it take me to ride up a very steep hill?

The questions were written on cards. Eight academically representative eight-year-olds and eight ten-year-olds were interviewed. Seventy-eight out of a possible 80 answers were generated by the children – only one child claimed that answers were impossible, and for two questions only.

The responses given to the bizarre questions indicated that the children attempt to make sense of what is presented to them. Often they draw on their own experiences:

To question 1 (Truck's load?)

Julie (10):... about over a tonne... it seems that a truck could take that much 'cause it's built for that kind of thing.

Bill (10):... about 80 pounds... I just guessed it. A truck down River Road, it's a big Kenworth, and it worked for a meat company.

To question 5 (Ride up steep hill?)

John (8):... about $\frac{1}{4}$ of an hour, or twenty minutes... well if it takes you ten minutes to ride five kilometres, then up a steep hill you'll be going slower... But *down* will be a lot faster.

Jason (10):... depends on what sort of bike... a 10-speed can zoom up a hill. Two minutes on a 10-speed. On a different kind of bike a bit longer. When I had my Cruiser it used to take me ages to climb hills.

With questions where there appeared to be the possibility of manipulating numbers, then children did so:

To question 2 (Time to dig hole and a half?)

Mike (8):... um... I wonder how big the hole is though?... 12 hours, half a day... if it takes one man to dig a hole, then two men $\frac{1}{2}$ of 24... I reckon a day or $\frac{1}{2}$ a day.

Rebecca (10):... 3 days... I added $1\frac{1}{2}$ and $1\frac{1}{2}$ together... I know $\frac{1}{2}$ and $\frac{1}{2}$ equals one, and one and one equals two. You add two and one.

To question 4 (How many sat test?)

Sue (8):... 11... fives into 55... five times 11 is 55... 'cause the top mark was 55, the bottom was 5.

Bruce (10):... About 15... well, usually most differences between people is about two or three percent, but usually it's greater than that, so about 15 sat the test.

In short, the children attempted to make sense of the situation in which they found themselves. This, of course, is exactly what they do each day in the mathematics lesson – for many students in our schooling system a considerable proportion of questions in mathematics must appear bizarre.

What, then, are the implications for teachers? How can we assist to construct appropriate meanings in mathematics?

Implications for Teachers

1. Children will not passively absorb what is presented to them. Children do not always learn that which the teacher intends. Keep this at the front of your mind while teaching. Explain concepts in different ways; build in regular maintenance; have realistic expectations for children; *listen* to children's explanations and questions.

2. We should assist children to take a greater responsibility for their learning. Children will attempt to make sense of even the most bizarre situations in mathematics when an *adult* is in charge. If children can be encouraged to become less dependent upon the teacher, then the knowledge becomes part of themselves.(This may be a difficult and different re-orientation for adults.)

3. Research shows that children can progress well in mathematics given the right environment. Such an environment includes: techniques that build upon existing knowledge, that involve the use of concrete materials (where needed), that promote discussion, that encourage children to ask questions, and interact with the teacher, and that reinforce important mathematical ideas (rather than polysyllabic labels).

4. We need to think critically about the materials we use. For example, the Form One (11-year-olds) textbook makes the following suggestion for slower children who need extra work on decimal fractions:

   Remedial:

   Those students who have difficulty with these pages may be asked to complete a pattern of multiplications in which the decimal point has different locations.

23×483	2.6×483	$.26 \times 483$
26×48.3	2.6×48.3	$.26 \times 48.3$
26×4.83	2.6×4.83	$.26 \times 4.83$

   Would the assigning of extra pencil-and-paper exercises such as these be of benefit to children struggling to cope with ideas behind operations on decimal fractions?

   Likewise, care must be taken when using apparatus and visual displays (such as number line models). Children may view the particular teaching aid in quite a different way from the teacher.

5. It is unwise to rely too heavily upon the spiral curriculum approach; do not put too much faith in the notion that if children don't master ideas and processes one year, they will pick up that knowledge the next. For many children the spiral curriculum has become two-dimensional, never rising above one level. As teachers we should aim for mastery of concepts and processes, realizing that for most children progress is gradual and slow. If children miss some key mathematical idea one year they may never gain that knowledge; when the idea is next confronted it is usually at a more abstract level, and even less attainable!

Notes

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The early work of Piaget in maths can be found in
Piaget, Jean (1941) *The Child's Conception of Number*. London: Routledge and Kegan Paul Limited.

Brueckner's testing to measure ability in compilation and verbal problems are described in
Brueckner, L.J. (1938) *Techniques of Diagnosis*. Chapter Eight in the 34th Yearbook of the National Society for the Study of Education. Chicago: University of Chicago Press.

The claim that it is mathematics education that is failing students, not students that are failing maths, can be found in
Fey, J.T. and Sonnabend, T. (1982) Trends in School Mathematics Performance. Chapter Six in Austin, G.R. and Gardner, J. (eds.), *The Rise and Fall of National Test Scores*. New York: Academic Press.
and
Whitney, Haasler (1984) *Taking Responsibility for School Mathematics Education*. Paper presented at I.C.M.E.V. Adelaide, Australia.

The British CSMS survey can be studied in
Hart, K.M. (1981) *Children's Understanding of Mathematics: 11–16*. London: John Murray.

The New Zealand IEA study is written up in
New Zealand Department of Education (1982) *The Second I.E.A. Mathematics Study*. Wellington: Education Department.

Studies in which children's mathematical ideas are probed through interviews are numerous. Two good articles by Erlwanger are mentioned below. See also
Davis, R.B. (1975) A Second Interview with Henry – Including some Suggested Categories of Mathematical Behaviour. *Journal of Children's Mathematical Behaviour*, Vol.1 (3), pp.36–62.
Davis, R.B. and McKnight, C. (1980) The Influence of Semantic Content on Algorithmic Behaviour. *Journal of Mathematical Behaviour*, Vol.3 (1), pp.39–87.
Alderman, D.L., Swinton, S.S. and Braswell, J.S. (1979) Assessing Basic Arithmetic Skills and Understanding Across Curricula. *Journal of Children's Mathematical Behaviour*, Vol.2 (2), pp.3–28.
Carr, K. (1983) Student Beliefs about Place Value and Decimals: Any Relevance for Science Education? *Research in Science Education*, Vol.13, pp.105–109.
Knight, G.H. (1982) *A Clinical Study of the Mathematical Incompetence of Some University Students*. Ph.D Thesis, Massey University.
Wearne, D. and Hiebert, J. (1984) *The Development of Meaning for Decimal Symbols*. Paper presented at I.C.M.E.V., Adelaide, Australia.

Studies in which interviews and data from large-scale surveys are combined include
Brown, M.L. (1981) *Levels of Understanding of Number Operations, Place Value and Decimals Among Secondary School Children*. Ph.D Thesis, University of London.
and
Brown, J.S. and Van Lehn, K. (1982) Towards a Generative Theory of Bugs. Chapter 9 in Carpenter T.P. *et al.*, *Additional Subtraction*. New Jersey: Lawrence Erlbaum.

Van Lehn's list of 77 'bugs' occurring in subtraction is in
Van Lehn, K. (1982) Bugs are not Enough: Empirical studies of Bugs, Impasses and Repairs in Procedural Skills. *Journal of Mathematical Behaviour*, Vol.3 (2), pp.3–72.

Studies displaying the capabilities of children:
Moser, J.M. and Carpenter, T.P. (1982) Young Children are Good Problem Solvers. *Arithmetic Teacher*, Vol.30 (3), pp.24–26.
Gelman, R. and Gallistel, C.R. (1983) *The Child's Understanding of Number*. Chapter 14 in Donaldson, M. (ed.), Early Childhood Development and Education. Oxford: Basil Blackwell.
Donaldson, M. (1978) *Children's Minds*. London: Fontana.

For the study on incompetent adults making good progress see
Duffin, J. (1978) Some Thought on numeracy. *Mathematics in School*, Vol.7 (5), pp.26–28.

That only 10% of 13-year-olds can use the 180° rule for triangles was revealed in
Carpenter, T.P., Matthews, W., Lindquist, M.M. and Silver, E.A. (1984) Achievement in Mathematics: Results from the National Assessment. *The Elementary School Journal*, Vol.84 (5), pp.485–496.

The quotation from Hart summarising the CSMS results is on p.209 of her book mentioned above.

Cockcroft's work with adults and the quotation can be found in
Cockcroft, W.H. (1982) *Mathematics Counts*. London: H.M.S.O.

Benny's conversations, and the other revealing misconceptions are in
Erlwanger, S.H. (1973) Benny's Conception of Rules and Answers in IPI Mathematics. *Journal of Children's Mathematical Behaviour*, Vol.1 (2), pp.7–26.
and
Erlwanger, S.H. (1975) Case Studies of Children's Conceptions of Mathematics – Part I. *Journal of Children's Mathematical Behaviour*, Vol.1 (3) pp.157–283.

The quotation on 2½-year-olds using the cardinal principle is from Gelman and Gallistell (p.198), mentioned above.

The work with 6-year-olds using advanced counting-on can be found in Moser and Carpenter mentioned above.

The full reference for Davis's book is
Davis, R.B. (1984) *Understanding Mathematics*. London: Croom Helm.

Is Red wider than Yellow? Such bizarre questions are asked in
Hughes, M. and Grieve, R. (1983) On Asking Children Bizarre Questions. Chapter 9 in Donaldson, M. (ed.), *Early Childhood Development and Education*. Oxford: Basil Blackwell.

The Form One text book quoted is
Duncan, E.R., Capps, L.R., Dolciani, M.P., Quast, W.G. and Zweng, M.J. (1980) *Modern School Mathematics: Structure and Use*. Boston: Houghton Mifflin Co.

That for some children the spiral curriculum in maths becomes two-dimensional is argued in Hart, mentioned above.

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