THE MEANING OF "EQUALS"

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Take a look at the following mathematics problem.

> Write the number in the square below that you think best completes the equation.
>
> 4 + 5 = □ + 3

How difficult do you consider this equation to be? At what age do you think a student should be able to complete it correctly?

I recently presented this problem to over 300 Year 7 and 8 students at a large intermediate school. More than half answered it incorrectly.[1] Moreover, the vast majority of the students who wrote something other than 6 were inclined to write the missing number as "9".

This result is not just an intermediate school phenomenon, however. Smaller numbers of students in other Years were also tested. In Years 4, 5 and 6, even larger percentages wrote incorrect responses, while at Years 9 and 10, more than 20 percent of the students were still not writing a "6" to complete the equation. Table 1 shows the overall picture.

TABLE 1: Percentages of Students Writing Different Responses at Years 7 and 8

Answer	*Year 7*	*Year 8*	*Total %*
9	49.2	35.2	45.2
6	40.7	60.7	47.5
12	6.8	0.0	3.9
Other	3.4	4.1	3.3
Total	100	100	100

When I talked to some of the students who had written "9" in the box, it soon became obvious that they had interpreted the equals sign as a signal to carry out the operation that precedes it. In the equation above, they saw the equals sign as a kind of command to complete the computation "5 + 4" and make an answer. For example:

> It's just a sign to say that the answer comes next.
>
> "Equals" is something to show you that you need to add up your answer.

This understanding of "equals" as meaning "work out the answer" was so strong that most of the students who got the question wrong had been prepared to ignore the "+ 3" part of the equation. A small minority, who were not happy leaving an extraneous "+ 3", put a "9" in the box, but then continued the equation by writing an additional equals sign and then a final answer of 12. This made what they considered to be a complete and accurate mathematical sentence.

Perhaps not surprisingly, many of the teachers of the classes who had been given the problem were quite perturbed by the low success rate of their students. Some had seen the equation as a completely trivial type of problem.

Before we jump to any conclusions about the quality of the maths programme or the mathematical ability of the students at the schools in the study, it is important to realise that this finding is not a new discovery in mathematics education. It reflects a common misconception about the equals sign amongst primary and intermediate school children that has been explored by mathematics education researchers internationally (Falkner, Levi and Carpenter, 1999; Kieran, 1981, 1992; MacGregor and Stacey, 1997; Saenz-Ludlow and Walgamuth, 1998; Stacey and MacGregor, 1999; Wright, 1999).

However, the fact that so many students appear to have this misconception highlights some important considerations that we, as mathematics educators, can learn from and begin to address.

In what follows I look at two of these considerations. I then go on to discuss some strategies we can employ in the classroom to help our students gain a richer sense of what the equals sign represents.

The meaning of "equals"

Firstly, it is worth considering what the equals sign means in a mathematical sense, and why so many children seem to struggle to develop this understanding.

From a mathematical point of view, the equals sign is not a command to do something. Rather it is a signifier of a very important relationship – that of equality.

Equality is all about sameness. An equation is a special mathematical design that allows the user to describe the relationship that exists between expressions of equal value. In fact, the relationship of equality that the equals sign defines is what allows equations to be manipulated and rearranged, so that new ways of expressing equality can be constructed and unknowns can be found. It makes equations the powerful tools that they are.[2,3]

Of course, there is nothing in the equals sign itself that imbues it with this mathematical meaning. It is a social convention that, over time, has been through a process of negotiation, modification and agreement. In fact, the pair of parallel lines we use today to signify that two expressions are equal has had a "bumpy" history. It is only in the last three hundred years or so that it has became generally recognised and used as the equals symbol (see box).

To have any meaning, the equals sign must be interpreted. Learners who come across the symbol have to construct its mathematical meaning for themselves. The mathematical quality of this interpretation or construction of meaning will depend on how they have experienced the equals sign and equations in the past. Unfortunately, it appears that for many primary and intermediate students, these past experiences have not (at least so far) led them to a rich understanding of what the equals sign really means.

It is probably not surprising that many children do develop a non-mathematical understanding or interpretation, especially when we consider their "everyday" experiences with the word "equals", and in particular with the equals sign.

Children often hear or see the word "equals" used in everyday language in a way that promotes a sense of causation. When we say something like, "Walking under a ladder equals bad luck," for example, we suggest that doing the first thing will lead to or cause the second.

Moreover, in school mathematics, especially when doing arithmetic, the "do the operation" meaning so often seems to fit. Students can quite happily answer questions such as 5 + 4 = □ by reading it as "5 plus 4 makes what?"

THE HISTORY OF THE EQUALS SIGN

The equals sign we employ today was first used to signify equality by an English mathematician, Robert Recorde, in a book he wrote about algebra called *The Whetstone of Witte*. Recorde wanted to avoid having to "tediously repeat" the term "is equal to", so employed two parallel lines as a kind of shortcut. His rationale for choosing to use two parallel lines was:

… bicause noe 2 thynges can be moare equalle.

At first the sign was used inconsistently, sometimes even as a kind of decimal point. It was only from about Shakespeare's time, especially after it was employed in the work of such mathematical heavyweights as Descartes and Newton, that the equals sign was adopted as the universal symbol to represent equality.

Even the calculator tends to reinforce this meaning, by using the equals sign to label the key that executes the compute command, and "makes" the answer.

The effect on learning in mathematics

So should we be concerned that so many students struggle with the meaning of the equals sign? The short answer is yes. We now turn to look at how a weak sense of the equals sign can and does affect learning in mathematics.

As has already been noted, the "make the answer" understanding of the equals sign seems to serve well enough when students are carrying out and recording any arithmetic or computation. However, if we want students to develop the kinds of understandings and habits of mind involved in algebra, then it is vital that they are given opportunities to develop a mathematically richer understanding of equality.

The research literature certainly emphasises this. Falkner (Falkner, Levi and Carpenter, 1999), for instance, comments that a non-mathematical sense of the equals sign is "one of the major stumbling blocks for students when they move from arithmetic to algebra" (p.234, inset).

It is worth examining why this could be. When we do an arithmetic problem, we are generally focused on the particular numbers and computational procedures that will lead directly to a solution. Our goal, according to Kieran (1992), is to "… find the answer, and this goal is usually accomplished by carrying out some sequence of arithmetic operations on either the given numbers of the problem, or on the intermediate values derived therefrom" (p.393).

Algebraic thinking, on the other hand, involves a completely different way of looking at a problem. In algebra, we are more interested in describing and exploiting the relationships that exist within the problem. Often an equation is developed to show how the elements in the problem are related; then this equation is manipulated and, if required, solved for particular values. Lesh (Lesh, Post and Behr, 1988) calls this a "describe-transform-compute procedure".

MacGregor (MacGregor and Stacey, 1999) defines the difference between the two ways of thinking like this: "The language of arithmetic is focused on answers. The language of algebra is focussed on relationships" [PAGE NUMBER?].

When we develop an understanding of the equals sign as signifying a special relationship that exists between the two sides of an equation, we gain an insight which allows us to go beyond simply operating on numbers to find answers. We move into the realm of algebra. We begin to recognise that the mathematical expressions and equations we write down or read are objects in their own right with their own meanings, and as such are tools with which we can reason.

Sfard (cited in Kieran, 1992) describes this as the difference between conceiving of mathematical notation structurally (as an object), or operationally (as a process). According to Sfard: "Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time" (p.392).

So gaining an insight into the mathematical meaning of the equals is part of a significant shift for students that opens up new ways to reason mathematically.[4]

However, when students continue to interpret the equals sign as simply a command to perform an operation or a series of operations in order to provide an answer, they are denied this insight. In particular, they are unable to reflect on the relationship described by an equation. As a result, activities that involve exploiting that relationship are reduced simply to following rote procedures. Mathematics becomes rule following, rather than sense making.

Developing a richer understanding

So how can we help our students to develop a richer mathematical sense of what "equals" means?

According to the literature, simply explaining what the sign means is not enough. Falkner notes: "A concerted effort over an extended period of time is required to establish appropriate notions of equality" (Falkner, Levi and Carpenter, 1999, p.233). These writers also emphasise that it is important to start early: "Teachers should also be concerned about children's conceptions of equality as soon as symbols for representing number operations are introduced" (p.233).

Perhaps central in our attempts to develop better understandings should be a deeper awareness of how we communicate with our students. The fact that such a key and apparently simple concept can have significantly different meanings for the participants in a mathematics class should alert us to how easy it is to "talk past each other". If we are not alert to the meanings and ideas our students have, we can easily make assumptions that are quite wrong or unhelpful. If we are to develop shared meanings, we must be involved in conversations with students, so that meanings can be created and negotiated.[5]

Below, drawing from the literature and my own experience, I have proposed several ideas that could be used in the classroom to help students focus on the meaning of the equals sign.

Exploring the equals sign and equations

One way to start is to make the equals sign itself the subject of a conversation. There are various ways to do this. For instance, the symbol itself could be examined – perhaps even its history explored. Children could invent alternative ways of signifying that two things are equal, and explain their reasoning.

Alternatively, equations themselves can be used as discussion pieces. When we present an equation such as $4 + 5 = \square + 3$ to a class or group, the different conceptions of the equals sign that emerge can provide the basis for a productive session where children's ideas about equality can be expressed and considered. Another similar type of equation that can elicit interesting conversations is in the form $264 + 14 = \square + 264$. Here there is a significant structural similarity between the two sides, which the equals sign suggests that we can exploit.

Conjectures and refutations

The discussions we have about equations also provide a great opportunity to involve the development and testing of conjectures and refutations. Present students with equations such as $\square + 2 = 5$ and ask them for any statements they can make about the value of the missing number, compared with 5. For instance, should it be bigger or smaller? With older children, the numbers we use in the equations could be bigger, or decimals or fractions could be used. Older students could also be given equations such as $\square = \bigcirc + 2$ and asked to discuss what can be said about the number represented by the square, compared with the number represented by the circle, if the equation is going to be true.

True and false number sentences

Falkner, Levi and Carpenter (1999) describe a variation of the conjectures and refutations approach that is more suitable for younger students. Building on the work of Robert Davis (Davis, 1994), they show how true and false number sentences can be used to elicit discussion, in order to help develop conceptions regarding the equals sign. Number sentences such as the following (some true and some false) can be presented to students, and the students asked whether the equations are true or false. Students can also be asked to "prove" the assertions that they make.

$3 + 2 = 5$	$12 – 8 = 5$	$7 = 5 + 2$
$8 + 2 = 10 + 4$	$7 + 4 = 15 – 4$	$6 = 6$

Varying our representation of equations

It is also important to vary how equations are represented. Instead of writing equations with the unknown as the answer, for instance in the form $3 + 4 = \square$, vary it so that students get to see the empty box presented in other positions, for example as $\square = 3 + 4$. Don't expect all students to accept this kind of variation at first. To start with, this kind of arrangement could be quite nonsensical to many of them, and result in some vigorous discussion.

Creating new names for numbers

Another activity that emphasises the meaning of the equals sign involves getting students to create different ways of naming or expressing a number, and using the equals sign to link them: for instance, $3 + 4 = 7 = 5 + 2 = 1 + 6$, etc.

MacGregor and Stacey (1999) describe a more practical version of this activity. They show how students could use a geoboard to investigate how many rectangles can be made with a piece of string 24 units long. An equals sign could then be used to connect all the expressions that describe the perimeter of each rectangle.

For instance: $24 = 4 \times 6 = 6 + 6 + 6 + 6 = 12 + 12 = 2 + 10 + 10 + 2$. Making the units on the geoboard some size other than one, for instance 0.2 or ½, could make this activity more suitable for older children.

The equation as a balance

Picturing equations as a kind of balance beam is another way to challenge students' ideas about the equals sign. Students can draw, imagine, or even use a pan balance to model how an equation works. To be even or equal, the balance must of course have the same value (or weight) on both sides. The American National Council of Teachers has some interesting computer simulations of the equation as a pan balance on the internet. Teachers are free to download and use these with students. The website is: http://illuminations.nctm.org/imath/across/balance/index.html

Using the equals sign in notation

The final idea involves challenging students about the way the equals sign is often used in their working: that is, as a way of linking a series of computations, for instance: $3 + 7 = 10 \div 2 = 5$. Kieran (1981, p.323) comments that this is often observed even at university level. In terms of conventional mathematics, it is inappropriate. When we notice students doing this, it could be a good opportunity to introduce and talk about other symbols that can be used to link steps, such as an arrow. Using an arrow, the example above could then be rewritten as:

$3 + 7 = 10 \longrightarrow 10 \div 2 = 5$.

Looking at the ways the equals sign is used in working is also an opportunity to talk about what rigour means in mathematics. Liping Ma (1999), who contrasts mathematics teaching in the United Sates and China, notes that using the equals sign as a linking symbol would never be accepted by Chinese mathematics teachers. She comments: "From the Chinese teachers' perspective … the semantics of mathematical operations should be represented rigorously. It is intolerable to have two different values on each side of an equals sign" (p.111).

To finish

There is definitely more to the equals sign than meets the eye. As we have seen, many students, even at Year 7 and 8, struggle to interpret its meaning. They will have to change their perspective if they are to be successful learners in algebra. We have also seen, however, that there are practical steps that we can take to challenge students' conceptions, and lead them towards a richer understanding.

Perhaps the biggest lesson we can gain from examining how the equals sign is interpreted is an insight into the way our development of algebraic thinking is so deeply connected to the experiences we have with arithmetic. While solving number problems, we notice patterns and properties, and begin to develop generalisations about the way numbers and notation work. These form the foundations for algebra.[6]

Primary and intermediate classrooms are central sites in children's development of number ideas. It is important, therefore, that they are also places where attention is brought to the patterns and properties that exist in the world of numbers, and that opportunities are provided to allow children to discuss and explore them. As we have seen, holding up the equals sign as a kind of artefact for children to examine and explore is one way of doing just this.

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Notes

1 The problem was presented to whole classes as a pencil and paper task in the form above. The students answered the question individually and then talked about the various answers that had been submitted. The problem was presented to roughly equal numbers of Year 7 and 8 students.

2 Liping Ma, comparing Chinese and American approaches to teaching elementary mathematics (Ma, 1999), makes this comment:
"As my elementary teacher once said to her class, 'The equals sign is the soul of mathematical operations.' In fact, changing one or both sides of an equals sign for certain purposes while preserving the "equals" relationship is the 'secret' of mathematical operations" (p.111).

3 Mathematicians sometimes talk about three special properties that belong to the equals sign. The first is called the reflexive property. This just says that anything is equal to itself. In other words, the equal sign is like a mirror: the image it "reflects" is the same as the original. The second property of equality is the symmetric property. This says that if a = b, then b = a, that is, the equals sign works both ways. Finally, there is the transitive property of equality. Here, if a = b and b = c, then a = c. While it might not be necessary for students to know these properties by name, they are certainly important ideas for students to understand and explore.

4 In many ways a student's realisation that the equals sign signifies a relationship mirrors the way that algebra itself has developed historically. At first algebraic expressions were simply a way to record or prescribe mathematical operations on numbers. The last three hundred years or so has seen a shift to exploring and manipulating the expressions themselves, and exploiting the relationships that exist between them.

5 Brent Davis (1996) outlines a radical understanding of mathematics teaching and learning. The idea of mathematics teaching as a conversation is central to his thinking.

6 Algebra is often called generalised arithmetic.

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